International Journal of Theoretical Physics, Vol. 44, No. 12, December 2005 (© 2005)
DOI: 10.1007/s10773-005-8022-2

Complete Axiomatizations for Quantum Actions

A. Baltag[1] and S. Smets[2,3]

We present two equivalent axiomatizations for a logic of quantum actions: one in terms of *quantum transition systems*, and the other in terms of *quantum dynamic algebras*. The main contribution of the paper is conceptual, offering a new view of quantum structures in terms of their underlying *logical dynamics*. We also prove Representation Theorems, showing these axiomatizations to be *complete with respect to the natural Hilbert-space semantics*. The advantages of this setting are many: (1) it provides a clear and intuitive dynamic-operational meaning to key postulates (e.g. Orthomodularity, Covering Law); (2) it reduces the complexity of the Solèr–Mayet axiomatization by replacing some of their key higher-order concepts (e.g. "automorphisms of the ortholattice") by first-order objects ("actions") in our structure; (3) it provides a link between traditional quantum logic and the needs of quantum computation.

KEY WORDS: dynamic quantum logic; quantum frames; quantum dynamic algebra; quantum transition systems; quantales; Piron lattices.

PACS: 02.10.-v Logic; set theory and algebra; 03.65.-w Quantum mechanics; 03.65.Fd Algebraic methods; 03.67.-a Quantum information.

1. INTRODUCTION

Our research is connected to the recent trend towards a "dynamification" of logic, development pursued (mainly, but not exclusively) by the "Dutch school" in modal logic, see e.g. (van Benthem, 1996): looking at various "propositional" logics as being about *actions*, rather than propositions. This is also connected to the older work (originating in Computer Science) on Propositional Dynamic Logic (PDL) and its relatives (such as Hoare logic). More generally, there is already a whole tradition in Computer Science of *thinking about information systems in a dynamic manner*: a "state" of a system is, in this view, identified only by the actions that can be (successfully) performed on the state. This view is embodied in the various semantic notions of "*process*" that have been proposed in the

[1] Oxford University Computing Laboratory, Oxford University, Oxford, United Kingdom.
[2] Vrije Universiteit Brussel, Brussel, Belgium.
[3] To whom correspondence should be addressed at Vrije Universiteit Brussel, Flanders' Fund for Scientific Research Post-Doc; e-mail: sonsmets@vub.ac.be.

Computer Science literature: labeled transition systems, automata, coalgebras etc. On the other hand, there exists a completely independent, but similar trend towards "dynamification" in the *quantum logic* community, trend started by (Daniel, 1982, 1989; Faure *et al.*, 1995), and more recently developed by the "Brussels school" in quantum logic, in a series of papers (Amira *et al.* 1998; Coecke *et al.*, 2001, 2004; Coecke and Smets, to appear; Coecke and Stubbe, 1999; Smets, to appear).

All our past and present work on quantum systems may be thought of as a *blending of these separate trends* and an *application of these ideas to the logic of quantum information*. Our starting point was the observation made by (Coecke *et al.*, 2001; Coecke and Smets to appear) and developed further in (Baltag and Smets, 2004; Coecke *et al.* (2004); Smets to appear) that the traditional *"propositional" quantum logic is already, in fact, an essentially dynamic logic*. This is reflected by the fact that the so-called "quantum implication" $\phi \stackrel{S}{\to} \psi$ (also called "Sasaki hook," and defined as $\sim (\phi \wedge \sim (\phi \wedge \psi))$, where $\wedge$ is conjunction, i.e. intersection, of quantum propositions, and $\sim$ is the orthocomplement) is in fact not a (deductive) implication at all, both for axiomatic[4] and semantic reasons: the semantics of the Sasaki hook is not given by an underlying consequence relation; instead, its most natural semantics is in terms of *measurements*: a system satisfies $\phi \stackrel{S}{\to} \psi$ if, after any successful measurement of property ϕ, the system will necessarily satisfy property ψ. But this is a *dynamic* notion: indeed, if we think of the successful measurement of ϕ as a "quantum test" action $\phi?$ (of testing property ϕ on a quantum system), Sasaki hook corresponds to the *dynamic modality* $[\phi?]\psi$ in Dynamic Logic (see e.g. Harel *et al.*, 2000), and defines what in Computer Science is called *the weakest precondition of action* $\phi?$ *with respect to (a postcondition)* ψ. In quantum logic, this dynamic view can be traced back to the analysis of the Sasaki hook as a Stalnaker conditional presented in Hardegree, (1975, 1979) and is reflected upon in (Beltrametti and Cassinelli, 1977; Smets, 2001).

As we will see, we take these dynamic modalities as the basic operators of our *quantum dynamic logic*. But once this step is taken, it is natural to extend this notion to *other kinds of physically meaningful actions, beyond measurements*; in particular, we can take weakest preconditions for *unitary evolutions* (the "quantum gates" of Quantum Computing); more generally, we can apply this to any *quantum "action,"* i.e. any action that can be obtained from quantum tests $\phi?$ and unitary evolutions U via the operations of *sequential composition* $\alpha \cdot \beta$ ("do first α then β") and *non-deterministic choice* $\cup_{i\in I}\alpha_i$ ("do either one of the actions α_i"). Quantum actions are an abstraction of the notion of physical action, and in particular of the notion of "program" used in Quantum Computation. The "choice" operation is needed here both to define real *measurements* (since

[4] It does not satisfy the Deduction Theorem, which is the main logical property to be expected of a deductive implication.

a measurement is a non-deterministic choice over a family of tests of mutually orthogonal properties) and *while*-programs (since the program *while* ϕ *do* α can be encoded as $\cup_{n\geq 0}(\phi? \cdot \alpha)^n \cdot (\neg\phi)?$).

In this paper, we propose *two equivalent axiomatizations of the "logic" of quantum actions*. The first is in terms of *Quantum Transition Systems* (which we also call *quantum dynamic frames*). This is a semantical abstraction of quantum systems, in the form of a *relational structure*, of the type known in Computer Science as "labeled transition systems," and in modal logic as "(multi-modal) Kripke frames." It takes as fundamental the notion of *(quantum) state* and *represents quantum actions as relations between states*. This is the well-known *input-output* view of programs, identifying them with their "behavior" on states. Similarly, a "property" in this view is nothing but a "set of states" (the ones satisfying the property). Our axiomatization in terms of quantum frames extends the older work on relational semantics for orthologic (based on orthoframes[5]) to the full (orthomodular) quantum logic, and beyond. One of the main problems of orthoframes (as shown in Goldblatt, 1984) was that orthomodularity could not be captured by any first-order frame condition. In contrast, in our setting, orthomodularity corresponds to a nice first-order frame condition, with a natural dynamic/operational interpretation.

The second axiomatization is in terms of *Quantum Dynamic Algebras*. It takes the notion of *quantum action* as the fundamental one, and directly axiomatizes the underlying algebra. In this approach, "properties" are nothing but special types of actions (namely, "tests" i.e. projectors) and hence will be identified with them. This view is obviously the closest one to the traditional algebraic approach to quantum logic, conceived as a logic of projectors. Due to the presence of the above operations, the algebra of quantum actions will necessarily be a *quantale*. This approach has its origin in the work by Coecke *et al.* (Amira *et al.*, 1998; Coecke *et al.*, 2001; Coecke and Stubbe, 1999), where quantale structures have been first used to capture the dynamics of quantum systems (including unitary evolutions in Coecke *et al.*, 2001).

There is in fact *a third axiomatic approach*, closely related to the first: namely, building *a finitary modal language describing quantum transition systems*, in the same way that (classical) Propositional Dynamic Logic (PDL) is a finitary language for labeled transition systems. This approach, which we call *Quantum Dynamic Logic*, is closer to a "logistic" view of quantum logic, as a propositional logic with a finitary syntax axiomatized by a finite set of inference rules. But this approach will of course involve a *dynamic* interpretation of the "logistic" view, and this will be reflected in the more complex syntax of *Quantum Dynamic Logic*: as in PDL, the syntax will have two sides, involving both *propositions* and *actions*.

[5] See (Goldblatt, 1974); also known as *preclusivity spaces* (Dalla Chiara *et al.*, 2004) or *orthogonality spaces* (Foulis and Randall, 1971), or (in its dual version) *similarity spaces*.

Due to lack of space and for reasons of simplicity, we do not develop this third, more syntactic, approach here, but we plan to write a separate paper on the issue.[6]

Nevertheless, we think that the semantic and algebraic axiomatizations presented here are an important contribution to the "dynamification" of (traditional) quantum logic. One important result is that this allows us to tackle the *open problem* stated in (Dalla Chiara *et al.*, 2004) "of finding a calculus, if any exists, that is sound and complete with respect to **H**", where **H** is taken to be to the class of Hilbert lattices.[7] As the discussion in (Dalla Chiara *et al.*, 2004) clearly points out, traditional *orthomodular quantum logic* does *not*[8] provide a complete axiomatization with respect to **H**. Moreover, the stronger lattice-theoretic axiomatization of C. Piron (based on the addition of the so-called "Covering Law" to orthomodular quantum logic) is not complete (with respect to **H**) either.[9] Also, the existing complete lattice-theoretic characterizations of **H** (based on the work of Piron, 1964; Amemiya and Araki, 1967; Mayet, 1998; Solèr, 1995) are *not* given in *first-order* logical terms, but they make an *essential* use of higher-order concepts,[10] and hence they do not seem directly translatable into a first-order logical calculus. We claim here that what is needed is a new dynamic-logical perspective, which takes "actions" seriously, as first-class objects of the logic (instead of refering to them indirectly as second-order concepts), thus allowing us to encode higher-order aspects into a first-order language (with action modalities).

In this context, the *Representation Theorem* proved in Section 3 of this paper, establishing the *completeness of our axiomatizations of the logic of quantum actions with respect to infinite-dimensional Hilbert spaces*, can be seen as a partial solution to the open problem in (Dalla Chiara *et al.*, 2004). Our proof is based on an extension of (Mayet's version of) Solèr's Theorem (Mayet, 1998; Solèr, 1995), itself an extension of Piron's Representation Theorem for Piron lattices (Piron, 1964, 1976; Amemiya and Araki, 1967). There is a technical condition (due to Mayet) that we impose, requiring the existence of a special unitary action, that is needed to ensure the existence of an orthonormal basis (for the underlying generalized Hilbert space).

A final remark: our quantum dynamic algebra could be seen as a partial vindication of the Jauch–Piron operational approach to quantum logic (Jauch, 1968; Jauch and Piron, 1969; Piron, 1964, 1976). Instead of constructing a framework on

[6] See also (Baltag and Smets, 2004), for a related finitary modal logic for compound quantum systems.

[7] A Hilbert lattice, as defined in (Dalla Chiara *et al.*, 2004), is any ortholattice based on the set of closed subspaces of some Hilbert space of dimension at least two.

[8] As a counter example, see the failure of the so-called orthoarguesian law.

[9] A *Piron lattice* is a lattice satisfying the conditions given in (Piron, 1976), i.e. it is an atomistic, irreducible, complete orthomodular lattice that satisfies the covering property. Every Hilbert lattice is a Piron lattice, but the converse is false: as shown by Keller (1980), this is due to the existence of Piron lattices based on "generalized Hilbert spaces" on non-arhimedian division rings.

[10] For example "automorphisms" of the given lattice.

all (equivalent) yes–no questions, we take tests as primitive. Tests can be viewed as specific yes–no questions, namely the ones corresponding to the "*filters*" in (Piron, 1976). One of the most basic and very often misunderstood ingredients (see e.g. Smets, 2003) of the Jauch–Piron approach is the construction of the *conjunction or meet of properties* via *product questions*, i.e. *non-deterministic choice* of "questions" (which is most often viewed as a form of disjunction or join). This point is elucidated in our paper: it is simply based on the fact that the weakest precondition of a union of actions is the conjunction of the weakest preconditions of each action. This allows us to follow Piron in our dynamic algebra and *define conjunction of properties via union ("choice") of actions* (using our "test for failure" $\sim \pi$ of an action π, which is the weakest precondition ensuring the impossibility of executing action π). This offers a physical justification, not only for conjunctions of incompatible properties, but even for the completeness of the lattice of properties: indeed, infinite unions of actions are physically meaningful, as non-deterministic choices between infinitely many alternative actions. For instance, a measurement with respect to an infinite orthornormal basis (or any infinite set of orthogonal states) is just such an infinite union of tests.

2. QUANTUM TRANSITION SYSTEMS

2.1. Dynamic Frames

For a given binary relation $R \subseteq \Sigma \times \Sigma$ on a set Σ, and subsets $S, P \subseteq \Sigma$, we define the *image of S via R*:

$$R(S) = \{t \in \Sigma : \exists s \in S\, (s, t) \in R\}$$

and the *weakest precondition of R with respect to (postcondition) P*:

$$[R]P = \{s \in \Sigma : \forall t \in \Sigma\ ((s, t) \in R \Rightarrow t \in P)\}$$

The second differs, in general, from the standard notion of preimage; but, when R is a *partial function*, the two notions coincide. The name "weakest precondition" comes from Computer Science: when we think of R as being the input–output relation of a program, then $[R]P$ captures the weakest condition that must be satisfied by the input-state to ensure that any output-state will satisfy condition P. In other words, we have:

$$S \subseteq [R]P \text{ iff } R(S) \subseteq P$$

When the set S is a singleton $S = \{s\}$ consisting of a single state, we skip the set notation and write $R(s)$ and $[R]s$ instead.

A *labeled transition system* (also called *(multi-modal) Kripke frame*) is a relational structure $(\Sigma, \{\stackrel{a}{\rightarrow}\}_{a \in \mathcal{A}})$, consisting of a set Σ of "*states*" and a family of "*transition relations*" $\overset{a}{\to} \subseteq \Sigma \times \Sigma$ between states, relations labeled by a set $\mathcal{A}$ called *basic actions*.

In PDL, one also considers some special kind of actions, called "*tests*." Each classical property $P \in \mathcal{P}(\Sigma)$ gives a "test" $P?$, which is nothing but the *diagonal* $\{(w, w) : w \in P\}$ of the set P. This could be thought of as a "purely epistemic" action by a classical (external) observer, "testing" property P, without in any way changing the state of the system. So the basic actions of PDL could be classified in two types: ("purely epistemic") tests $P?$, on one hand, and on the other (non-testing, "purely dynamic") basic actions U.

We slightly generalize this setting by defining:

A *dynamic frame* is a labeled transition system $\mathcal{F} = (\Sigma, \{\overset{P?}{\to}\}_{P \in \mathcal{L}}, \{\overset{U}{\to}\}_{U \in \mathcal{U}})$, consisting of:

1. a set Σ of objects, called *states*;
2. a family of binary "transition" relations $\overset{P?}{\to} \subseteq \Sigma \times \Sigma$, labeled by "test" actions $P?$; the action labels come from a given family $\mathcal{L} \subseteq \mathcal{P}(\Sigma)$ of subsets $P \subseteq \Sigma$, called *testable properties*;
3. a family of binary "transition" relations $\overset{U}{\to} \subseteq \Sigma \times \Sigma$, labeled by basic "actions" $U \in \mathcal{U}$, which we will call *basic unitary evolutions*.[11]

We can equip any dynamic frame $\mathcal{F}$ with a *measurement relation*:

$$s \to t \text{ iff } s \overset{P?}{\to} t \text{ for some } P \in \mathcal{L}$$

Intuitively, this means that state t can be obtained from state s by performing a measurement. The negation of this gives an *orthogonality relation* on states:

$$s \perp t \text{ iff } s \not\to t$$

Also, for any set $S \subseteq \Sigma$, we write $t \perp S$ iff we have $t \perp s$ *for all* $s \in S$. Moreover, we define the *orthogonal* (or *orthocomplement*) of the set S by $\sim S := \{t \in \Sigma : t \perp S\}$. The *biorthogonal closure of a set* S is simply the set $\sim\sim S = \sim (\sim S)$. It is easy to see that we always have $S \subseteq \sim\sim S$. When we actually have equality $S = \sim\sim S$, then we say that the set S is *biorthogonally closed*. Finally, for basic actions $\alpha \in \mathcal{L} \cup \mathcal{U}$ and testable properties $P \in \mathcal{L}$, we use the abbreviations: $\alpha(P) := (\overset{\alpha}{\to})(P)$, and $[\alpha]P := [\overset{\alpha}{\to}]P$, for the image and the weakest precondition of the set P via the relation $\overset{\alpha}{\to}$.

Example 1. *Kripke frames for classical PDL* are a special case of dynamic frames, in which one takes $\mathcal{L} =: \mathcal{P}(\Sigma)$, and the transition relation for a test is given

[11] We call them "basic" since they might not be closed under composition; in a Hilbert space, the composition of two unitary maps is unitary, but here it is possible to think of $\mathcal{U}$ as consisting of only *some* basic logic gates, from which all the "relevant" ones can be generated by composition.

by: $s \stackrel{P?}{\to} t$ iff $s = t \in P$; i.e. it is simply the diagonal of the set P (while the transitions $\stackrel{U}{\to}$ are arbitrarily chosen binary relations on Σ). The "measurement relation" is simply the identity $s = t$, and thus two states are "orthogonal" iff they are distinct, so $\sim S = \Sigma \setminus S$ is the complement and $\sim\sim S = S$.

This example shows that classical PDL encodes a *classical notion of measurement: every set $P \subseteq \Sigma$ is considered a "testable" property, and successful tests do not change the current state*. As mentioned above, PDL tests are classical (non-interactive, "purely epistemic") observations. It is now natural to consider a *quantum version* of dynamic frames, in which measurements are non-classical,[12] and so there exist no "purely epistemic" actions, every observation involving a change of state. Σ is then the set of states of a physical system, orthogonality is non-trivial, the testable properties are the biorthogonally closed subsets of Σ, and unitary evolutions are the "purely dynamic" actions.

2.2. Quantum Frames

A *quantum transition system* (or *quantum dynamic frame*) is a dynamic frame $\mathcal{F} = (\Sigma, \{\stackrel{P?}{\to}\}_{P \in \mathcal{L}}, \{\stackrel{U}{\to}\}_{U \in \mathcal{U}})$, satisfying the following list of conditions (in which variables P, Q range over testable properties in $\mathcal{L}$, variables s, t, s', t', v, w range over states in Σ and U ranges over evolutions):

1. Closure under *arbitrary conjunctions*: if $\mathcal{L}' \subseteq \mathcal{L}$ then $\bigcap \mathcal{L}' \in \mathcal{L}$
2. *Atomicity*. States are testable, i.e. $\{s\} \in \mathcal{L}$.
   This is equivalent to requiring that "states can be distinguished by tests," i.e. if $s \neq t$ then $\exists P \in \mathcal{L}: \ s \perp P, \ t \not\perp P$
3. *Adequacy*. Testing a true property does not change the state:
   if $s \in P$ then $s \stackrel{P?}{\to} s$
4. *Repeatability*. Any property holds after it has been successfully tested:
   if $s \stackrel{P?}{\to} t$ then $t \in P$
5. "*Covering Law*." If $s \stackrel{P?}{\to} w \neq t \in P$, then there exists some $v \in P$ such that $t \to v \not\to s$.
6. *Self-Adjointness Axiom*: if $s \stackrel{P?}{\to} w \to t$ then there exists some element $v \in \Sigma$ such that $t \stackrel{P?}{\to} v \to s$
7. *Proper Superposition Axiom*. Every two states of a quantum system can be properly superposed into a new state: $\forall s, t \in \Sigma \exists w \in \Sigma \ s \to w \to t$
8. *Reversibility and Totality Axioms*. Basic unitary evolutions are total bijective functions: $\forall t \in \Sigma \exists ! s \ s \stackrel{U}{\to} t$ and $\forall s \in \Sigma \exists ! t \ s \stackrel{U}{\to} t$

[12] Note that the non-classical measurements we have in mind are the ideal quantum measurements of the first kind introduced in (Pauli, 1980).

9. *Orthogonality Preservation*. Basic unitary evolutions preserve (non) orthogonality: Let $s, t, s', t' \in \Sigma$ be such that $s \stackrel{U}{\rightarrow} s'$and $t \stackrel{U}{\rightarrow} t'$.
   Then: $s \rightarrow t$ iff $s' \rightarrow t'$.
10. *Mayet's Condition: Orthogonal Fixed Points*. There exists some unitary evolution $U \in \mathcal{U}$ and some property $P \in \mathcal{L}$, such that U maps P into a proper subset of itself; and moreover the set of fixed-point states of U has dimension ≥ 2. In other words:
    $\exists U \in \mathcal{U} \exists P \in \mathcal{L} \exists t, w \in \Sigma \forall s \in \sim\sim \{t, w\} : \quad U(P) \subseteq P, U(P) \neq P, t \perp w, U(s) = s.$

Note: Our axioms imply that $\mathcal{L}$, with set-inclusion as partial order, forms a *Piron lattice* of infinite height: indeed, axiom 7 above ("Proper Superposition") implies the irreducibility of the lattice; axiom 5 is equivalent (in the context of the other axioms) to the usual statement of Covering Law for $\mathcal{L}$, and it also implies that "test" relations $\stackrel{P?}{\rightarrow}$ are partial functions (i.e. the outcome of a test is unique). Similarly, axiom 6 (Self-Adjointness) is equivalent (in the context of the other axioms) to the Orthomodularity Law for $\mathcal{L}$; finally, axiom 10 implies the existence of infinitely many mutually orthogonal states. Observe that all the above notions are defined in *purely dynamical terms* (e.g. the definition of orthogonality $s \perp t$ above simply means the impossibility of performing any test P? on state s and reach state t). Consequently, our axioms above provide *dynamic/operational interpretations* to all the postulates of a Piron lattice (as well as to the Solèr-Mayet postulates). For instance, we think that axioms 5 and 6 give a new and fresh insight into *the operational meaning of the Covering Law and Orthomodularity Law.*

Main Example: Concrete Quantum Dynamic Frames. Any classical Hilbert space[13] $\mathcal{H}$ can be structured as a quantum dynamic frame $\mathcal{F}(\mathcal{H})$, by taking: as set of "states" Σ, the family of all *one-dimensional closed linear subspaces of* $\mathcal{H}$; as class of testable properties $\mathcal{L}$, the family of *closed linear subspaces of* $\mathcal{H}$; as "test" actions P?, the maps induced on Σ by the projectors on the closed subspace associated to each corresponding $P \in \mathcal{L}$; as the set of "basic unitary actions" $\mathcal{U}$, the family of all (maps induced on Σ by) unitary operators on $\mathcal{H}$. Any subframe of $\mathcal{F}(\mathcal{H})$ satisfying the above conditions is called a *concrete quantum dynamic frame*. As we will prove later, every quantum dynamic frame is isomorphic to a concrete quantum dynamic frame.

2.3. Quantum Actions Over a Frame

Given a quantum dynamic frame $\mathcal{F}$, the *class of quantum actions* (or *quantum programs*) over $\mathcal{F}$ is defined as the smallest family of binary relations

[13] That is a Hilbert space over one of the following fields: reals, complex numbers or quaternions.

$\mathcal{Q} \subseteq \mathcal{P}(\Sigma \times \Sigma)$ which contains all tests $\{\stackrel{P?}{\rightarrow}\}_{P\in\mathcal{L}}$, and all basic unitary evolutions $\{\stackrel{U}{\rightarrow}\}_{U\in\mathcal{U}}$ as well as their inverses $\stackrel{U^{-1}}{\rightarrow} := (\stackrel{U}{\rightarrow})^{-1} = \stackrel{U}{\leftarrow}$, and is closed under the operations of *relational composition*[14] $R \cdot R'$ and *arbitrary union*[15] *of relations* $\cup_{i\in I} R_i$. We denote by $\mathcal{Q}(\mathcal{F})$ the family of all quantum actions over $\mathcal{F}$.

Intuitively, relational composition represents *sequential composition* of actions: *do first action* π *then action* π'; while arbitrary union gives us *non-deterministic choice*: *do either one of the actions* $\{\pi_i\}_{i\in I}$. An action π, that can be expressed *without the use of choice* $\cup$ (i.e. only as a sequential composition of tests and basic unitary actions) is called *deterministic*. As relations, such actions are *partial functions*, i.e. for a given input-state s, they have *at most one output-state* t (such that $s \stackrel{\pi}{\rightarrow} t$).

Proposition 1. *(Adjointness for quantum actions)*
Let $\mathcal{D}$ be the family of deterministic actions.

- There exists a unique map$^{\dagger} : \mathcal{D} \rightarrow \mathcal{D}$, satisfying the conditions: $(P?)^{\dagger} = P?$, $U^{\dagger} = U^{-1}$, $(U^{-1})^{\dagger} = U$, $(\pi \cdot \sigma)^{\dagger} = \sigma^{\dagger} \cdot \pi^{\dagger}$.
  We call $\pi^{\dagger}$ the *adjoint* of the action π.
- Let $s, w, t \in \Sigma$: If $s \stackrel{\pi}{\rightarrow} w \rightarrow t$ then there exists some element $v \in \Sigma$ such that $t \stackrel{\pi^{\dagger}}{\rightarrow} v \rightarrow s$.

Proposition 2. *(Weakest Precondition)*
For any quantum action $\pi \in \mathcal{Q}(F)$ and any testable property $P \in \mathcal{L}$, we have the following:

- the weakest precondition $[\pi]P$ is a testable property, i.e. $[\pi]P \in \mathcal{L}$;
- the *kernel* of any quantum action is a testable property, i.e.:
  $\mathrm{Ker}\,[\pi] = [\pi]\emptyset \in \mathcal{L}$
- $\mathrm{Ker}\,(P?) = \sim P$

Notations: The last claim suggests an extension of the orthocomplement notation to quantum actions, by putting:

$$\sim \pi := \mathrm{Ker}\,(\pi) = [\pi]\emptyset$$

We will later use this notion in our axiomatization of quantum dynamic algebras.[16] We also denote by

$$\pi[P] :=\sim\sim \pi(P)$$

the biorthogonal closure of the image of P via π.

[14] Relational composition is defined by: $(s, t) \in R \cdot R'$ iff $\exists w\, (s, w) \in R \wedge (w, t) \in R'$.
[15] defined by: $(s, t) \in \bigcup_{i\in I} R_i$ iff $\exists i \in I\, (s, t) \in R_i$.
[16] But note that in general we have $\sim\sim \pi \neq \pi$. In fact, $\sim\sim \pi = \pi$ iff π is a test!

Proposition 3. *(Strongest Testable Postcondition)*
For any quantum action $\pi \in \mathcal{Q}(\mathcal{F})$ and any testable property $P \in \mathcal{L}$, we have the following:

- $\pi[P] \in \mathcal{L}$
- $\pi[P]$ is *the strongest testable postcondition* ensured by executing π on any state satisfying (precondition) P; i.e., for all $Q \in \mathcal{L}$: $\pi[P] \subseteq Q$ iff $\pi(P) \subseteq Q$
- For *deterministic actions* $\pi \in \mathcal{D}$, we have $\pi(P) = \pi[P] = \sim [\pi^{\dagger}] \sim P$
- As a consequence, for deterministic actions π, we have a Galois duality between $[\pi]$ and $\sim [\pi^{\dagger}] \sim$; i.e. for all $S \subseteq \Sigma$: $S \subseteq [\pi]P$ iff $\sim [\pi^{\dagger}] \sim S \subseteq P$
- In particular, for tests $Q?$, we have: $S \subseteq [Q?]P$ iff $\sim [Q?] \sim S \subseteq P$

Recall that our dynamic modality $[Q?]P$ captures the meaning of the so-called Sasaki hook $Q \stackrel{S}{\rightarrow} P$, and so the strongest post-condition $Q?[P] = \sim [Q?] \sim P$ is just the ortho-dual of the Sasaki hook, i.e. the Sasaki projector. So the last claim in the above Proposition captures the well-known Galois duality between the Sasaki hook and Sasaki projector. Moreover, in the context of our other axioms, this last claim is equivalent to the Self-Adjointness Axiom, and thus to the Orthomodularity Law.[17]

3. QUANTUM DYNAMIC ALGEBRA

As the work of D. Kozen and V. R. Pratt in the late seventies and early eighties indicates, an algebraic semantics for a propositional dynamic logic can be presented as a dynamic algebra of actions. We will now extend this idea, linking it both to the work of Coecke *et al.* (2001) on using quantales to capture the dynamics of quantum systems, and to the work of Piron (1964), Solèr (1995), Mayet (1998) and others on the complete axiomatization of the lattice of complete subspaces of a Hilbert space. We obtain *Quantum Dynamic Algebras*, as a *complete axiomatization of the algebra of quantum actions*. But unlike (Solèr, 1995; Mayet, 1998), our axioms are *"essentially first-order,"* i.e. do not essentially[18] involve any quantification over high-order objects, such as automorphisms of the structure we describe.

Since we take *only actions* as objects in our algebra, we will have to identify "properties" p with their tests (projectors) $p?$. So there is no need for the "test" sign ?, and we can simply denote with p both the property and the action of testing

[17] For the equivalence between orthomodularity and this Galois duality, see e.g. (Coecke and Smets, to appear).

[18] We write "essentially," since the definitions of a "quantale" and of "the sub-quantale generated by a subset" involve trivial second-order quantifications over all subsets of a given structure.

it. So now the weakest precondition becomes an operation *on actions*. Since this is not a very natural action operation, we take instead as basic a special case of it, namely the operation

$$\pi \ \longmapsto \ \sim \pi \ := \mathrm{Ker}\,(\pi) = [\pi]\emptyset$$

mapping an action to its kernel. The kernel, as a property, expresses the impossibility of executing action π, and the test $(\sim \pi)?$ of this property can be understood as a "*test for failure*" of the action π. Since we identify the property with its test, $\sim \pi$ is the same as $(\sim \pi)?$, and thus we can simply interpret this as an operation taking any action π to the action $\sim \pi$ of "testing for failure of action π."

We will be able to recover the (test of) the weakest precondition $[\pi]p$ as $\sim (\pi \cdot \sim p)$. More importantly, we can recover the *lattice operations on $\mathcal{L}$ by defining them only in terms of action operations*: indeed, a conjunction $\bigwedge_{i\in I} p_i$ is true iff we have $\sim \bigcup_{i\in I} \sim p_i$, i.e. none of the failure tests $\sim p_i$ can be executed.

A *Quantum Dynamic Algebra* is a structure:

$$(\mathcal{Q}, \bigcup, \cdot, , \sim)$$

satisfying a number of axioms (to follow). We call the elements of $\mathcal{Q}$ *quantum actions* (or programs), and use variables $x, y, \ldots$ to denote them. The types of our operations are as follows: the *union* (or "*choice*") $\bigcup : \mathcal{P}(\mathcal{Q}) \to \mathcal{Q}$ is an infinitary operation, *(sequential) composition* $\cdot : \mathcal{Q} \times \mathcal{Q} \to \mathcal{Q}$ is a binary operation, and the "*test for failure*" (of an action) $\sim: \mathcal{Q} \to \mathcal{Q}$ is a unary operation. The actions of the form $\sim x$ are called *tests*, or "*properties*," and they can be thought of as an abstraction of the notion of projector (or closed linear subspace). We put

$$\mathcal{L} := \{\sim x : x \in \mathcal{Q}\}$$

for the set of all tests, and for convenience we use variables $p, q, \ldots$ to denote them (although all such variables can obviously be eliminated by replacing them with $\sim x$ etc.).

We make the following definitions and abstractions:

$0 := \bigcup \emptyset$	$1 :=\sim 0$
$[x]p :=\sim (x \cdot \sim p)$	$\bigwedge_i p_i :=\sim \bigcup_i \sim p_i$
$p \le q$ iff $p \wedge q = p$	$p \perp q$ iff $p \le\sim q$
$\bigvee_i p_i :=\sim\sim \bigcup_i p_i$	$At(\mathcal{L}) := \{p \in \mathcal{L} \mid \forall q \in \mathcal{L}(0 \ne q \le p \Rightarrow q = p)\}$

The last is the set of all *atoms* of $\mathcal{L}$. We also put

$$\mathcal{U} := \{x \in \mathcal{Q} : \exists y \ x \cdot y = y \cdot x = 1 \quad \text{and} \quad \forall p \in \mathcal{L} \ x \cdot \sim p = \sim (x \cdot p)\}$$

for the set of *all unitary evolutions*. We use variables (again, eliminable) u to denote the elements of $\mathcal{U}$. Observe that $\mathcal{U}$ is closed under composition and inverses. As usually, we denoted by u^{-1} the inverse of any $u \in \mathcal{U}$ (i.e. the unique element s.t.

$u \cdot u^{-1} = 1$). We also define the set $\mathcal{D}$ of *deterministic actions* by:

$$\mathcal{D} := \{x \in \mathcal{Q} : \forall a \in At(\mathcal{L}) \exists b \in At(\mathcal{L}) a \leq [x]b\}$$

Note that $\mathcal{L} \cup \mathcal{U} \subseteq \mathcal{D}$ and that $\mathcal{D}$ is closed under composition. For any action $x \in \mathcal{D}$ and any property $p \in \mathcal{L}$, we define the *image-set of p via x* by:

$$x(p) := \{b \in At(\mathcal{L}) : \exists a \in At(\mathcal{L}) \exists y \in \mathcal{D}\, a \leq p, y \subseteq x, a \leq [y]b, a \not\leq \sim y\}$$

The *strongest post-condition* internalizes this image-set inside $\mathcal{L}$, by taking its (quantum) join:

$$x[p] := \bigvee x(p)$$

This is an element of $\mathcal{L}$ which represents the (test corresponding to the) biorthogonal closure of the image. But, for *deterministic* actions, this closure coincides with the image. Moreover, we have:

$$x \text{ deterministic}, a \in At(\mathcal{L}) \implies x[a] \in At(\mathcal{L}) \cup \{0\}$$

Axioms for Quantum Dynamic Algebras

The structure $(\mathcal{Q}, \bigcup, \cdot, , \sim)$ is required to satisfy the following conditions:

1. $0 \in \mathcal{L}$; or equivalently, $\sim 1 = 0$
2. $(\mathcal{Q}, \bigcup, \cdot, 1)$ is a quantale[19] generated by the set $\mathcal{L} \bigcup \mathcal{U}$ of tests and unitary evolutions.
3. *Choice*: $[\bigcup_i x_i]p = \bigwedge_i [x_i]p$; or, equivalently: $\sim \bigcup_i x_i = \bigwedge_i \sim x_i$
4. *Composition*: $[\pi \cdot \sigma]p = [\pi][\sigma]p$; or, equivalently: $\sim (x \cdot y) = [x] \sim y$
5. *Adequacy* : $p \wedge q \leq [q]p$, and also $p \wedge [p]q \leq q$
6. *Proper Superpositions*: if $p, q \neq 0$ then there exists $r \in \mathcal{L}$ such that $r \not\perp p, r \not\perp q$.
7. *Self-Adjointness Axiom*: For all $p, q \in \mathcal{L}$: $p \leq [q] \sim [q] \sim p$
8. "*Covering Law*": if $q \in At(\mathcal{L})$ and $p \not\perp q$, then $p \wedge (\sim p \vee q) \in At(\mathcal{L})$.
9. "*Atomicity*": $\mathcal{L}$ is *atomistic*, i.e.: $p \leq \bigvee \{q \in At(\mathcal{L}) : q \leq p\}$
10. *Mayet's Condition*. There exist $p, q, r \in \mathcal{L}$, $u \in \mathcal{U}$, such that for every $s \leq q \vee r$:
    $p \leq [u]p,\ p \neq [u]p,\ q, r \neq 0, q \perp r,\ s = [u]s$
11. *Actions are determined by their behavior on atoms*:
    If $x(a) = y(a)$ for all $a \in At(\mathcal{L})$, then $x = y$.
12. *Image commutes with unions*: $(\bigcup_{i \in I} x_i)(p) = \bigcup_{i \in I} x_i(p)$.
    (Note that here, $\bigcup$ on the left is the quantale *sup*, while $\bigcup$ on the right is set-theoretical union.)

[19] That is $(\mathcal{Q}, \bigcup)$ is a complete lattice, $(\mathcal{Q}, \cdot, 1)$ is a monoid and $\cdot$ distributes over $\bigcup$.

The statement of the last two axioms 11 and 12 may look set-theoretical, but these two axioms can be replaced by only one non-set-theoretical axiom:

11′. For all deterministic actions $x, x_i \in \mathcal{D}$:

$$x \subseteq \bigcup_{i \in I} x_i \text{ iff } \forall a \in At(\mathcal{L}) \exists i \in I\ x[a] \leq x_i[a]$$

where $\subseteq$ on the left is just the partial order relation[20] of the quantale $\mathcal{Q}$.

Example: Frame Algebras and Concrete Algebras

The quantum actions $\mathcal{Q}(\mathcal{F})$ over any quantum dynamic frame form a quantum dynamic algebra. In particular, the algebra $\mathcal{Q}(\mathcal{F})$ of quantum actions over a concrete frame $\mathcal{F} \subseteq \mathcal{F}(\mathcal{H})$, based on a classical Hilbert space $\mathcal{H}$, is called a *concrete quantum dynamic algebra*. Notice that, for an operator x in a concrete algebra, $\sim x$ is the (projector over the subspace given by the) *kernel* Ker(x) of the relation x.

Theorem 1. *Every quantum dynamic algebra is isomorphic to a concrete quantum dynamic algebra. As a consequence, every quantum dynamic frame is isomorphic to a concrete quantum dynamic frame.*

Proof: (sketch): It is easy to see that, in a quantum dynamic algebra, $\mathcal{L}$ forms a Piron lattice of infinite height. By Piron's theorem, there exists an isomorphism $i : \mathcal{L} \rightarrow \mathcal{L}(H)$ between $\mathcal{L}$ and the lattice of projectors $\mathcal{L}(H)$ over a generalized Hilbert space (also called orthomodular space) $\mathcal{H}$. Also, any basic unitary element $u \in \mathcal{U}$ induces an automorphism of the ortholattice $\mathcal{L}$, given by:

$$p \mapsto u[p] = [u^{-1}]p$$

(where u^{-1} is the inverse of $u \in \mathcal{U}$). This, together with our Axiom 10 above, ensures that the special unitary element (call it u_0) from Axiom 10 satisfies the conditions of Lemma 1 in (Mayet, 1998); by applying this lemma, we conclude this special element u_0 is induced by a unique *unitary operator* (in the sense of Hilbert spaces!) $j(u_0)$ on the underlying generalized Hilbert space $\mathcal{H}$, i.e. we have $j(u_0)(i(p)) = i(u_0[p])$. This, together with (the other clauses included in) our Axiom 10 above, ensures that the lattice $\mathcal{L}$ fulfills Mayet's condition in (Mayet, 1998), equivalent to Solèr's condition in (Solèr, 1995), i.e. to the existence in $\mathcal{H}$ of an infinite set of orthonormal vectors. By Solèr's theorem, $\mathcal{H}$ must be a classical Hilbert space, so $\mathcal{L}$ is isomorphic to the lattice of projectors (or closed linear subspaces) of a classical Hilbert space. It is easy to see that this implies that *every*

[20] Note that the restriction of this quantale order $\subseteq$ to $\mathcal{L}$ is identity, and thus it does *not* coincide with the order $\leq$ of the lattice $\mathcal{L}$.

automorphism of $\mathcal{L}$ is induced by a unique unitary operator on $\mathcal{H}$: so *all* our unitary elements $u \in \mathcal{U}$ are induced by (uniquely determined) unitary operators $j(u)$ on $\mathcal{H}$. Let now $\overline{A}$ be the map induced on "states" (one-dimensional subspaces) in $\mathcal{F}(\mathcal{H})$ by any operator A on $\mathcal{H}$. We define a map $I : \mathcal{Q} \rightarrow \mathcal{Q}(F(\mathcal{H}))$, by putting: $I(p) = \overline{i(p)}$, $I(u) = \overline{j(u)}$, $I(x \cdot y) = I(x) \cdot I(y)$ and $I(\bigcup_{i \in I} x_i) = \bigcup_{i \in I} x_i$. The last two axioms of our algebra ensure that this is a well-defined unique map. Together with the other axioms, they ensure that it is in fact an embedding. □

The Representation Theorem for quantum frames $\mathcal{F}$ follows from this, together with the fact that $\mathcal{Q}(\mathcal{F})$ is a quantum dynamic algebra, whose atoms correspond to the states of the frame.

4. CONCLUSION

We presented two axiomatic structures for the logic of quantum actions. The power of our approach comes from the underlying "(inter)action"-based philosophy. As explained above, this stays closely connected to the operational approach initiated by Jauch and Piron. In particular, a true characterization of a physical system should be based on the structure of the actual and potential physical qualities of the system itself, which encode how the system might act and react under all relevant circumstances. Indeed, the axiomatization presented in this paper replaces the known static approach to quantum logic with a straightfoward representation (as a labeled transition system) of the potential dynamic behavior of the physical system itself.

As mentioned above, with our quantum dynamic frames comes naturally (as with any Kripke frame) an associated modal logic, in this case, a *quantum version of dynamic logic* (PDL). Our representation theorem yields a *completeness result for this logic*, which we plan to present in a future paper. This could be considered as a positive solution to the open problem in (Dalla Chiara, 2004) "of finding a calculus, if any exists, that is sound and complete with respect to (the class of Hilbert lattices) **H**." Moreover, the dynamic character of our logic makes it relevant to Quantum Computation: extensions of our work to multi-partite systems (such as in Baltag and Smets, 2004) could play the same role in analyzing quantum programs (and proving their correctness) that classical PDL (and Hoare logic) played for classical programs.

However, much work remains to be done in this sense. For instance, in (Baltag and Smets, 2004) we extended the logic of quantum actions to cover *entanglements* (in addition to tests and evolutions), obtaining a logical calculus for entangled systems, which we called the *Logic of Quantum Programs* (LQP). But to date, we have no completeness results for LQP, and even the existence of any such complete calculus for entanglement is an open problem.

Another problem, of equal importance for quantum computation, is extending our setting to deal with the *quantitative aspects of quantum information*, and in particular with notions like "*phase*" and "*probability*.' Our aim in this paper was to develop a logic to reason about *qualitative* quantum information flow, so we neglected the *probabilistic* aspects of quantum systems. There are natural ways to extend our setting, using the notion of *probabilistic transition system*, and we plan to investigate them in future work.

We hope that this paper will provide a new significant contribution (in addition to other related work (Abramsky and Coecke; Baltag and Smets, 2004) to the on-going search for a *complete "logical calculus of quantum information flow*."

ACKNOWLEDGMENTS

We thank Samson Abramsky, Bob Coecke, and Isar Stubbe for their useful comments on the material presented in this paper.

REFERENCES

Abramsky, S. and Coecke, B. (2004). A categorial semantics of quantum protocols. In *Proceedings of the 19th annual IEEE symposium on Logic (LICS) in Computer Science*. IEEE Computer Society, 2004. Available at arXiv:quant-ph/0402130.

Amemiya, I. and Araki, H. (1967). A remark on Piron's paper. *Publications of the Research Institute of Mathematical Sciences, Kyoto University, Series A* **2**, 423–427.

Amira, H., Coecke, B., and Stubbe, I. (1998). How quantales emerge by introducing induction within the operational approach. *Helvetica Physica Acta* **71**, 554–572.

Baltag, A. and Smets, S. (2004). The logic of quantum programs. In *Proceedings of the 2nd International Workshop on Quantum Programming Languages (QPL2004), TUCS General Publication*, P. Selinger, ed., Turku Center for Computer Science, Finland, vol. 33, pp. 39–56.

Beltrametti, E. G. and Cassinelli, G. (1977). On state transformations induced by yes-no experiments, in the context of quantum logic. *Journal of Philosophical Logic* **6**, 369–379.

Coecke, B. and Smets, S. (to appear). The Sasaki hook is not a [static] implicative connective but induces a backward [in time] dynamic one that assigns causes. *International Journal of Theoretical Physics*. Available at arXiv: quant-ph/0111076.

Coecke, B. and Stubbe, I. (1999). On a duality of quantales emerging from an operational resolution. *International Journal of Theoretical Physics* **38**, 3269–3281.

Coecke, B., Moore, D. J., and Smets, S. (2004). Logic of dynamics & dynamics of logic; some paradigm examples. *Logic, Epistemology and the Unity of Science*, S. Rahman, J. Symons, D. M. Gabbay, and J. P. Van Bendegem, eds., Kluwer Academic, Dordrecht.

Coecke, B., Moore, D. J., and Stubbe, I. (2001). Quantaloids describing causation and propagation for physical properties. *Foundations of Physics Letters* **14**, 357–367. Available at (arXiv: quant-ph/0009100).

Dalla Chiara, M., Giuntini, R., and Greechie, R. (2004). *Reasoning in Quantum Theory, Sharp and Unsharp Quantum Logics*, Kluwer Academic, Dordrecht .

Daniel, W. (1982). On the non-unitary evolution of quantum systems. *Helvetica Physica Acta* **55**, 330–338.

Daniel, W. (1989). Axiomatic description of irreversible and reversible evolution of a physical system. *Helvetica Physica Acta* **62**, 941–968.

Faure, C. L.-A., Moore, D. J., and Piron, C. (1995). Deterministic evolutions and schrodinger flows. *Helvetica Physica Acta* **68**, 150–157.

Foulis, D. J. and Randall, C. H. (1971). Lexicographic orthogonality. *Journal of Combinatorial Theory* **11**(2), 157–162.

Goldblatt, R. (1974). Semantic analysis of orthologic. *Journal of Philosophical Logic* **3**, 19–35.

Goldblatt, R. (1984). Orthomodularity is not elementary. *The Journal of Symbolic Logic* **49**, 401–404.

Hardegree, G. M. (1975). Stalnaker conditional and quantum logic. *Journal of Philosophical Logic* **4**, 399–421.

Hardegree, G. M. (1979). The conditional in abstract and concrete quantum logic. *The Logico-Algebraic Approach to Quantum Mechanics*, C. A. Hooker, ed., Vol. 2, D. Reidel, Dordrecht.

Harel, D., Kozen, D., and Tiuryn, J. (2000). *Dynamic Logic*, MIT-Press, Massachusetts.

Jauch, J. M. (1968). *Foundations of Quantum Mechanics*, Addison-Wesley, Reading, Massachusetts.

Jauch, J. M. and Piron, C. (1969). On the structure of quantal proposition systems. *Helvetica Physica Acta* **42**, 842–848.

Keller, H. A. (1980). Ein nichtklassischer hilbertscher Raum. *Mathematische Zeitschrift* **172**, 41–49.

Mayet, R. (1998). Some characterizations of the underlying division ring of a Hilbert lattice by automorphisms. *International Journal of Theoretical Physics* **37**(1), 109–114.

Pauli, W. (1980). *Handbuch der Physik, Vol. 5, Part 1: Prinzipien der Quantentheorie 1 (1958)*; English translation by P. Achuthan and K. Venkatsesan: *General Principles of Quantum Mechanics*, Springer Verlag, Berlin.

Piron, C. (1964). Axiomatique quantique (PhD-Thesis), *Helvetica Physica Acta*, **37**, 439–468, English Translation by M. Cole: “Quantum Axiomatics” RB4 Technical memo 107/106/104, GPO Engineering Department (London).

Piron, C. (1976). *Foundations of Quantum Physics*, W.A. Benjamin, Massachusetts.

Smets, S. (2001). On Causation and a counterfactual in quantum logic: The Sasaki hook. *Logique et Analyse* **173–175**, 307–325.

Smets, S. (2003). In defense of operational quantum logic. *Logic and Logical Philosophy* **11**, 191–212.

Smets, S. (to appear). From intuitionistic logic to dynamic operational quantum logic. *Poznan Studies in Philosophy and the Humanities*.

Solèr, M. P. (1995). Characterization of Hilbert spaces by orthomodular spaces. *Communications in Algebra* **23**(1), 219–243.

van Benthem, J. (1996). *Exploring Logical Dynamics*, Studies in Logic, Language and Information, CSLI Publications, Stanford.